Combining Set Variable Representations

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Abstract. Within constraint programming, a number of different representations have been used for set variables. Many of these representations are based on combining together existing representations. To understand such combinations, we provide a formal definition of this notion of combination, and of the synchronisation between representations. We characterize two types of combinations, define bound consistency on these combinations and provide some tractability results. Finally, we compare the strength of the different combinations, positioning existing representations within our taxonomy. This systematic approach opens the door to interesting new representations of set variables. Our results apply also to multiset variable representations.

1 Introduction

Set variables are a useful modelling and solving construct found in some of the earliest constraint solvers [12, 4]. A representation for set variables used in many solvers are subset bounds: a lower bound containing the set of definite elements and an upper bound containing the set of possible elements. However, this is a weak approximation and several new and stronger representations which better approximate domains have therefore been proposed. Many of these new representations are based on combining together existing representations (e.g. [1, 5, 14, 11]). For instance, the Cardinal set solver [1] reasons simultaneously about subset and cardinality bounds on the set. Similar observations hold for the multiset variable representations. In [9], for instance, cardinality and variety bounds are incorporated to the subset bounds of multiset variables [7].

Such previous work has differed on how the representations are combined, how consistency between the representations is maintained, and how constraints are propagated. Unfortunately no formal framework exists to compare these combined representations. Our goal is to provide such a framework. We use this framework to launch a systematic study of representations. Due to lack of space, we here focus only on set variables but our results are directly applicable to multiset representations and their combinations. Our study provides insight into existing combined representations, as well as new and potentially useful representations. Finally, it uncovers a number of important relations. For example, we prove that the *lengthlex* representation for set variables (which combines lexicographic ordering and cardinality information and is designed to exploit cardinality and ordering constraints) is incomparable to a representation for set variables that deal with the lexicographic ordering and cardinality information separately or simultaneously. As a second example, we prove that the hybrid representation of [14] (which combines lexicographic, subset and cardinality bounds) is tighter than one that looks at pairs of bounds at the same time, but looser than one that looks at all the bounds simultaneously.

2 Background

In this paper, we consider finite domains and only set variables. A set variable takes as value a subset of a universe $U = \{1, \ldots, n\}$. The domain $D(S)$ for a set variable S can potentially contain up to 2^n values. The representation R of a set domain is therefore usually given by some bounds that limit the domain to subsets of 2^U that form an interval in some partial or total order on 2^U. The approximation $D_R(S)$ of a domain $D(S)$ is described by its lower bound $lb_R(S)$ and its upper bound $ub_R(S)$. If $D_R(S)$ is the smallest interval containing $D(S)$, we call it the *tightest* approximation of $D(S)$.

Our first representation for set variables uses the subset partial order [12, 4]. The elements in lb are the definite elements, and the elements in $ub \setminus lb$ are the possible elements. This is the *subsetbounds* (SB) representation. More formally, given a variable S, $D_{SB}(S) = \{s \mid lb_{SB}(s) \subseteq s \subseteq ub_{SB}(s)\}$. Given a set of sets $A \subseteq 2^U$, we define $glb_{SB}(A) = \bigcap_{a \in A} a$. Similarly we define $lub_{SB}(A) = \bigcup_{a \in A} a$.

Another approach is to define a total ordering $\leq_T$ on 2^U and to represent the domain through a lower bound lb_T and an upper bound ub_T. That is, in the representation T, $D_T(S) = \{s \mid lb_T(S) \leq_T s \leq_T ub_T(S)\}$. This is the case in [5, 14] where the sets are ordered lexicographically, though in two different ways. We here call *minlex* (also denoted by ML) the total ordering in which sets are ordered by $\leq_{ML}$ defined as follows: elements in the sets are ordered from smallest to largest and the lists obtained in this way are ordered lexicographically. Given a variable S, $lb_{ML}(S)$ and $ub_{ML}(S)$ represent the smallest and the greatest sets (wrt $\leq_{ML}$) in $D_{ML}(S)$. *lengthlex* (also denoted by LL, [5]) is a representation in which sets are ordered according to $\leq_{LL}$, that is, by increasing size, and ties are broken with $\leq_{ML}$. $lb_{LL}(S)$ and $ub_{LL}(S)$ represent the smallest and the greatest sets (wrt $\leq_{LL}$) in $D_{LL}(S)$. The hybrid domain from [14] orders elements in the sets from greatest to the smallest element as opposed to *minlex*, and then orders lexicographically the resulting lists. We refer to this ordering as *maxlex*. Given a set of sets $A \subseteq 2^U$ and a representation T defined by a total ordering $\leq_T$, we define $glb_T(A) = \min_{\leq_T}(A)$. Similarly we define $lub_T(A) = \max_{\leq_T}(A)$.

A third approach is based on a measure on the size or value of a set. We illustrate this with the cardinality representation (*card*) but we could consider other similar representations based on the cost of the set. In *card*, $D_C(S) = \{s \mid lb_C(S) \leq |s| \leq ub_C(S)\}$. Given a set of sets $A \subseteq 2^U$, we define $glb_C(A) = \min(\{|a| \mid a \in A\})$. Similarly we define $lub_C(A) = \max(\{|a| \mid a \in A\})$.

We now give a uniform definition of bound consistency (BC) on any of these representations. Given some representation R, variables $S_1, \ldots, S_n$ and a constraint $c(S_1, \ldots, S_n)$, we denote by $c[S_i]_R$ the set $\{s_i \mid c(s_1, \ldots, s_i, \ldots, s_n) \wedge s_j \in D_R(S_j) \forall j \in 1..n\}$.

Definition 1 (Bound consistency on R). *Given a representation R, variables $S_1, S_2, \ldots, S_n$, and a constraint $c(S_1 \ldots, S_n)$, we say that S_i is bound consistent on c for R iff $lb_R(S_i) = glb_R(c[S_i]_R)$ and $ub_R(S_i) = lub_R(c[S_i]_R)$. $c(S_1 \ldots, S_n)$ is bound consistent for R if S_i is bound consistent on c for R for $i \in 1..n$.*

When a constraint $c(S_1 \ldots, S_n)$ is not bound consistent, enforcing bound consistency on c is a way to reduce the search space to explore. Enforcing bound consistency on c consists of increasing $lb_R(S_i)$ and decreasing $ub_R(S_i), i \in 1..n$ until bound consistency holds without throwing out solutions of c. When no such lower and upper bounds exist (i.e., when $D_R(S_1) \times \ldots \times D_R(S_n)$ does not contain any solution of c), we say that bound consistency has failed on c.

Example 1. Consider two variables S_1 and S_2 in universe $U = \{1, 2, 3\}$ represented using $R = minlex$ with $lb_{ML}(S_1) = lb_{ML}(S_2) = \{\}$ and $ub_{ML}(S_1) = ub_{ML}(S_2) = \{3\}$ and the constraint $c \equiv |S_1 \cap S_2| \leq 2$. We have $c[S_1]_{ML} = \{\{1, 2\}, \{1, 2, 3\}, \{1, 3\}, \{2\}, \{2, 3\}\}$, hence BC on *minlex* would update $lb_{ML}(S_1)$ to $glb_{ML}(\{\{1, 2\}, \{1, 2, 3\}, \{1, 3\}, \{2\}, \{2, 3\}, \{3\}\})$ which is $\{1, 2\}$, and $ub_{ML}(S_1)$ to $lub_{ML}(\{\{1, 2\}, \{1, 2, 3\}, \{1, 3\}, \{2\}, \{2, 3\}\})$ which is $\{2, 3\}$. Similarly for S_2.

3 Combinations of Two Representations

One way to have a tighter representation for a set variable is to combine several representations by maintaining upper and lower bounds simultaneously on two different orderings.

3.1 Synchronization of bounds

With using two representations simultaneously, a desirable property is that both representations agree on their bounds. We call this property *synchronization* of the bounds. Given two representations R_1 and R_2, we write $D_{R_1R_2}(S)$ for $D_{R_1}(S) \cap D_{R_2}(S)$. We will say that a representation R_1 is synchronized with a representation R_2 on a set variable S when $lb_{R_1}(S)$ and $ub_{R_1}(S)$ are consistent with $D_{R_1R_2}(S)$. We say that S is synchronized for R_1 and R_2 iff R_1 and R_2 are synchronized with each other.

Definition 2 (Synchronization). *Given a variable S, and two representations R_1 and R_2, then R_1 is synchronized with the representation R_2 on S iff:*

- $lb_{R_1}(S) = glb_{R_1}(D_{R_1R_2}(S))$;
- $ub_{R_1}(S) = lub_{R_1}(D_{R_1R_2}(S))$;

S is synchronized for R_1 and R_2 iff R_1 is synchronized with R_2 and R_2 is synchronized with R_1.

Example 2. Consider a variable S and universe $U = \{1, 2, 3\}$ represented using $R_1 = minlex$ with $lb_{ML}(S) = \{1, 2\}$ and $ub_{ML}(S) = \{3\}$, and using $R_2 = subsetbounds$ with $lb_{SB}(S) = \{\}$ and $ub_{SB}(S) = \{1, 3\}$. We have $D_{ML}(S) = \{\{1, 2\}, \{1, 2, 3\}, \{1, 3\}, \{2\}, \{2, 3\}, \{3\}\}$ and $D_{SB}(S) = \{\{\}, \{1\}, \{3\}, \{1, 3\}\}$. The intersecting domain $D_{MLSB}(S) = \{\{1, 3\}, \{3\}\}$. For *minlex* to be synchnorised with *subsetbounds*, its lower bound $lb_{ML}(S)$ must be $glb_{ML}(D_{MLSB}(S)) = \{1, 3\}$. Similarly, for *subsetbounds* to be synchnorised with *minlex*, its lower bound $lb_{SB}(S)$ must be $glb_{SB}(D_{MLSB}(S)) = \{3\}$.

3.2 Bound consistency on pairs of representations

We consider two different ways to propagate a combined representation. In the *weak* sense, we apply bound consistency on each representation independently and synchronize them. In the *strong* sense, we ensure the lower and upper bounds of a representation are bound consistent on their synchronized domains. Given two representations R_1 and R_2, the weak combination will be denoted by R_1+R_2 and the strong combination by $R_1 \times R_2$.

Definition 3 (Bound consistency on $R_1 + R_2$). *Given variables $S_1, \ldots, S_n$, and a constraint $c(S_1 \ldots, S_n)$, S_i is bound consistent on c for R_1+R_2 iff S_i is synchronized for R_1 and R_2, S_i is BC on c for R_1 and S_i is BC on c for R_2.*

Example 3. Consider a variable S and universe $U = \{1, 2, 3\}$ represented using $R_1 = minlex$ with $lb_{ML}(S) = \{1\}$ and $ub_{ML}(S) = \{3\}$, and using $R_2 = subsetbounds$ with $lb_{SB}(S) = \{1\}$ and $ub_{SB}(S) = \{1, 2, 3\}$, and the constraint $c \equiv |S| = 2$. The variable S is BC on c for SB but is not for *minlex*. Enforcing BC on c for *minlex* would update the bounds of *minlex* so that their cardinality is 2, hence we get $lb_{ML}(S) = \{1, 2\}$ and $ub_{ML}(S) = \{2, 3\}$. With these domains however, S is not synchronised for *minlex* and SB, because SB requires that 1 belongs to S. Ensuring synchronisation, $ub_{ML}(S)$ is further updated to $\{1, 3\}$. Now, S is BC on c for $minlex + SB$.

Definition 4 (Bound consistency on $R_1 \times R_2$). *Given variables $S_1, \ldots, S_n$ and a constraint $c(S_1 \ldots, S_n)$, S_i is bound consistent on c from R_1 to R_2 iff:*

- $lb_{R_1}(S) = glb_{R_1}(c[S_i]_{R_1 R_2})$;
- $ub_{R_1}(S) = lub_{R_1}(c[S_i]_{R_1 R_2})$;

S_i is bound consistent on c for $R_1 \times R_2$ iff S_i is synchronized for R_1 and R_2 and S_i is bound consistent from R_1 to R_2 and vice versa.

Example 4. Consider two variables S_1 and S_2 and universe $U = \{1, 2, 3\}$ represented using $R_1 = minlex$ with $lb_{ML}(S_1) = lb_{ML}(S_2) = \{1\}$ and $ub_{ML}(S_1) = ub_{ML}(S_2) = \{3\}$, and using $R_2 = card$ with $lb_C(S_1) = lb_C(S_2) = ub_C(S_1) = ub_C(S_2) = \{1\}$, and the constraint $c \equiv \{2, 3\} \subseteq S_1 \cup S_2$. Both S_1 and S_2 are synchronised for *minlex* and *card*, and they are both BC on c for both representations. Hence, S_1 and S_2 are BC on c for $minlex + card$. However, S_1 and S_2 are not BC on c for $minlex \times card$. We have $c[S_1]_{MLC} = \{\{2\}, \{3\}\}$, hence BC on $minlex \times card$ would update $lb_{ML}(S_1)$ to $glb_{ML}(\{\{2\}, \{3\}\})$ which is $\{2\}$. Similarly for S_2.

4 Comparison of Pairwise Combinations

To compare the pruning power of the (combined) representations, we define the *stronger* relation. Informally, one representation is stronger than another if bound consistency reasoning with the first representation is more powerful than with the second.

Definition 5 (Stronger relation $\succeq$). *Given two representations R_1 and R_2 and a constraint $c(S_1, \ldots, S_n)$, we say that R_1 is stronger than R_2 on c ($R_1 \succeq_c R_2$) iff $\forall D(S_i)$ with $D_{R_1}(S_i)$ and $D_{R_2}(S_i)$ the tightest approximation of $D(S_i)$ for R_1 and R_2, if $D_{R_2}(S_1) \times \ldots \times D_{R_2}(S_n)$ does not contain solutions of c then $D_{R_1}(S_1) \times \ldots \times D_{R_1}(S_n)$ does not contain solutions of c. (That is, if BC fails on c for R_2 then BC fails on c for R_1.)*

We say that $R_1 \succeq R_2$ iff $R_1 \succeq_c R_2$ for all constraints c. We say that R_1 is strictly stronger than R_2 on c ($R_1 \succ_c R_2$) iff $R_1 \succeq_c R_2$ and $R_2 \not\succeq_c R_1$. We say that R_1 is strictly stronger than R_2 ($R_1 \succ R_2$) iff $R_1 \succeq R_2$ and $R_2 \not\succeq R_1$. We say that R_1 and R_2 are equivalent on c ($R_1 \equiv_c R_2$) iff $R_1 \succeq_c R_2$ and $R_2 \succeq_c R_1$. We say that R_1 and R_2 are equivalent ($R_1 \equiv R_2$) iff $R_1 \succeq R_2$ and $R_2 \succeq R_1$. We say that R_1 and R_2 are incomparable on c ($R_1 \sim_c R_2$) iff $R_1 \not\succeq_c R_2$ and $R_2 \not\succeq_c R_1$. We say that R_1 and R_2 are incomparable ($R_1 \sim R_2$) iff $R_1 \not\succeq R_2$ and $R_2 \not\succeq R_1$.

To show that $R_1 \succeq R_2$, we could alternatively start from a domain $D_{R_2}(S_i)$ for R_2, construct the tightest approximated domain $D_{R_1}(S_i)$ of $D_{R_2}(S_i)$ for R_1, propagate bounds consistency on the given constraint in a way that R_1 fails but not R_2. Suppose that R_2 does not fail. This is because the bounds of $D_{R_2}(S_i)$ are supported. Since $D_{R_1}(S_i)$ is constructed from $D_{R_2}(S_i)$, it necessarily contains all sets that are in $D_{R_2}(S_i)$, hence the same supports exist for the same bounds.

Definitions for comparing strength of propagation are sometimes based on the values pruned. See, for example, [3]. We chose to define the strength of representations on their ability to detect failure because we can then compare different representations of domains that do not contain the same values. This choice is similar to that made in [2], where different representations for integer variables are compared.

The following theorems are useful to compare two (combined) representations.

Theorem 1. *Given two representations R_1 and R_2:*

- $(R_1 + R_2) \succeq R_1$ *and* $(R_1 + R_2) \succeq R_2$

– $(R_1 \times R_2) \succeq R_1$ *and* $(R_1 \times R_2) \succeq R_2$.

Proof. Immediate from the definitions. □

Theorem 2. *Given two representations R_1 and R_2, $(R_1 \times R_2) \succeq (R_1 + R_2)$.*

Proof. Let $c(S_1, \ldots, S_n)$ be a constraint. Suppose BC does not fail on c for $R_1 \times R_2$. Definition 4 tells us that BC on $R_1 \times R_2$ requires that for every S_i, $lb_{R_1}(S_i) = glb_{R_1}(c[S_i]_{R_1R_2})$, $lb_{R_2}(S_i) = glb_{R_2}(c[S_i]_{R_1R_2})$, $ub_{R_1}(S_i) = lub_{R_1}(c[S_i]_{R_1R_2})$, and $ub_{R_2}(S_i) = lub_{R_2}(c[S_i]_{R_1R_2})$. These bounds are synchronized because $c[S_i]_{R_1R_2} \subseteq D(S_i)_{R_1R_2}$. These bounds are also obviously BC for R_1 and BC for R_2, so BC for $R_1 + R_2$. Therefore, if BC does not fail for $R_1 \times R_2$, then $R_1 + R_2$ cannot fail, as in the worst case, it will converge on the same bounds as for $R_1 \times R_2$. □

5 Using the Framework

To illustrate the potential of this framework, we consider pairwise combinations of three of the most popular representations for sets: *SB*, *card* and *minlex*.

5.1 Synchronisation and BC

The synchronization property of Definition 2 gives rise to the following synchronisation rules between every pair of *minlex*, *SB*, and *card* representations.

Representations minlex and card

- $lb_C(S) \leq |lb_{ML}(S)| \leq ub_C(S)$ and $lb_C(S) \leq |ub_{ML}(S)| \leq ub_C(S)$;
- $lb_C(S) \in \{|s| \mid s \in D_{ML}(S)\}$ and $ub_C(S) \in \{|s| \mid s \in D_{ML}(S)\}$.

Representations minlex and subsetbounds

- $lb_{SB}(S) \subseteq lb_{ML}(S) \subseteq ub_{SB}(S)$ and $lb_{SB}(S) \subseteq ub_{ML}(S) \subseteq ub_{SB}(S)$;
- $lb_{SB}(S) \supseteq \cap_{s \in D_{ML}(S)}\{s\}$ and $ub_{SB}(S) \subseteq \cup_{s \in D_{ML}(S)}\{s\}$.

Representations subsetbounds and card

- $lb_C(S) = |ub_{SB}(S)| \rightarrow lb_{SB}(S) = ub_{SB}(S)$;
- $ub_C(S) = |lb_{SB}(S)| \rightarrow ub_{SB}(S) = lb_{SB}(S)$;
- $lb_C(S) \geq |lb_{SB}(S)|$, $ub_C(S) \leq |ub_{SB}(S)|$.

All these synchronization rules are polynomial to enforce. However, synchronization is not a trivial property. Indeed, it is intractable to achieve in general.

Theorem 3 (Complexity of Synchronization). *Given a variable S and two representations R_1 and R_2, enforcing synchronization of S for R_1 and R_2 is NP-hard even if R_1 and R_2 are defined by total orderings for which computing the successor of an element is polynomial.*

Proof. The proof is omitted for space reasons. □

Corresponding BC rules instead follow immediately Definition 4.

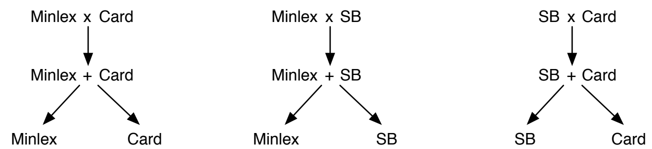

Fig. 1. Pairwise comparison of representations $minlex, SB$ and $card$. $H_1 \rightarrow H_2$ means $H_1 \succ H_2$.

5.2 Pairwise Comparison

We undertook a pairwise comparison of $minlex$, SB, and $card$ and their combinations. Results are summarised in Figure 1. Similar results hold for $maxlex$ instead of $minlex$.

Theorem 4 (Comparing $minlex$ and $card$ combinations).
1) $(minlex \times card) \succ (minlex + card)$
2) $(minlex + card) \succ minlex$
3) $(minlex + card) \succ card$

Proof. By Theorem 1 and Theorem 2, $(minlex \times card) \succeq (minlex + card)$, $(minlex + card) \succeq minlex$ and $(minlex + card) \succeq card$.

To show that $(minlex \times card) \succ (minlex + card)$, take the constraint $S_1 \cup S_2 \supseteq \{1, 3\}$ and universe $U = \{1, 2, 3\}$. Assume that $D(S_1) = D(S_2) = \{\{1\}, \{2\}\}$. Then $minlex$ and $card$ domains are: $D_{ML}(S_1) = D_{ML}(S_2) = \{\{1\}, \{1, 2\}, \{1, 2, 3\}, \{1, 3\}, \{2\}\}$; $lb_C(S_1) = ub_C(S_1) = lb_C(S_2) = ub_C(S_2) = 1$. BC on $minlex \times card$ fails whilst BC on $minlex + card$ does not prune.

To show that $(minlex + card) \succ minlex$, take the constraint $S_1 \cup S_2 \supseteq \{1, 2, 3\}$ and universe $U = \{1, 2, 3\}$. Assume that $D(S_1) = D(S_2) = \{\{1\}, \{2\}\}$. Then $minlex$ and $card$ domains are: $D_{ML}(S_1) = D_{ML}(S_2) = \{\{1\}, \{1, 2\}, \{1, 2, 3\}, \{1, 3\}, \{2\}\}$; $lb_C(S_1) = ub_C(S_1) = lb_C(S_2) = ub_C(S_2) = 1$. BC on $minlex + card$ fails whilst BC on $minlex$ does not prune.

Finally, to show that $(minlex + card) \succ card$, take the constraint $S \geq_{lex} \{2\}$ and universe $U = \{1, 2, 3\}$. Assume that $D(S) = \{\{1\}, \{1, 2\}\}$. Then $minlex$ and $card$ domains are: $D_{ML}(S) = \{\{1\}, \{1, 2\}\}$; $lb_C(S) = 1$ and $ub_C(S) = 2$. BC on $minlex + card$ fails since the constraint has no solution in the $minlex$ domain. However BC on $card$ alone does not prune. □

Theorem 5 (Comparing $minlex$ and SB combinations).
1) $(minlex \times SB) \succ (minlex + SB)$
2) $(minlex + SB) \succ minlex$
3) $(minlex + SB) \succ SB$

Proof. By Theorem 1 and Theorem 2, $(minlex \times SB) \succeq (minlex + SB)$, $(minlex + SB) \succeq minlex$ and $(minlex + SB) \succeq SB$.

To show that $(minlex \times SB) \succ (minlex + SB)$, take the constraint $|S1| + |S2| = 5$ and universe $U = \{1, 2, 3, 4, 5\}$. Assume that $D(S_1) = D(S_2) =$

$\{\{1,5\},\{3\}\}$. The *minlex* and *SB* domains are: $D_{ML}(S_1) = D_{ML}(S_2) = \{\{1,5\}, \{2\},\{2,3\},\{2,3,4\},\{2,3,4,5\},\{2,3,5\},\{2,4\},\{2,4,5\},\{2,5\},\{3\}\}$; $lb_{SB}(S_1) = lb_{SB}(S_2) = \{\}$, $ub_{SB}(S_1) = ub_{SB}(S_2) = \{1,3,5\}$. BC on $minlex \times SB$ fails since the constraint has no solution in the joint domain whilst BC on $minlex + SB$ does not prune.

To show that $(minlex + SB) \succ minlex$, take the disjoint constraint $S_1 \oplus S_2$ and universe $U = \{1,2,3\}$. Assume that $D(S_1) = D(S_2) = \{\{1,3\},\{3\}\}$. Then *minlex* and *SB* domains are: $D_{ML}(S_1) = D_{ML}(S_2) = \{\{1,3\},\{2\},\{2,3\},\{3\}\}$; $lb_{SB}(S_1) = lb_{SB}(S_2) = \{3\}$, $ub_{SB}(S_1) = ub_{SB}(S_2) = \{1,3\}$. BC on $minlex + SB$ fails whilst BC on *minlex* alone does not prune.

Finally, to show that $(minlex + SB) \succ SB$, take the constraint $S \geq_{lex} \{1,4\}$ and the universe $U = \{1,2,3,4\}$. Assume that $D(S) = \{\{\},\{1\},\{1,2\},\{1,2,4\}\}$. Then *minlex* and *SB* domains are: $D_{ML}(S) = \{\{\},\{1\},\{1,2\},\{1,2,3\},\{1,2,3,4\}, \{1,2,4\}\}$; $lb_{SB}(S) = \{\}$ and $ub_{SB}(S) = \{1,2,4\}$. BC on $minlex + SB$ fails since the constraint has no solution in the *minlex* domain. However BC on *SB* alone does not prune. □

Theorem 6 (Comparing SB and $card$ combinations).
1) $(SB \times card) \succ (SB + card)$
2) $(SB + card) \succ SB$
3) $(SB + card) \succ card$

Proof. By Theorem 1 and Theorem 2, $(SB \times card) \succeq (SB + card)$, $(SB + card) \succeq SB$ and $(SB + card) \succeq card$.

To show that $(SB \times card) \succ (SB + card)$, take the constraint $S_1 \cup S_2 \supseteq \{2,3,4\}$ and universe $U = \{1,2,3,4\}$. Assume that $D(S_1) = D(S_2) = \{\{1,2\},\{1,3\}, \{1,4\}\}$. The *SB* and *card* domains are $lb_{SB}(S_1) = lb_{SB}(S_2) = \{1\}$ and $ub_{SB}(S_1) = ub_{SB}(S_2) = \{1,2,3,4\}$; $lb_C(S_1) = ub_C(S_1) = lb_C(S_2) = ub_C(S_2) = 2$. BC on $SB \times card$ fails whereas BC on $SB + card$ does not prune.

To show that $(SB + card) \succ SB$, take the constraint $S_1 \cup S_2 \supseteq \{1,2,3\}$ and universe $U = \{1,2,3\}$. Assume that $D(S_1) = D(S_2) = \{\{1\},\{2\},\{3\}\}$. Then *SB* and *card* domains are: $lb_{SB}(S_1) = lb_{SB}(S_2) = \{\}$ and $ub_{SB}(S_1) = ub_{SB}(S_2) = \{1,2,3\}$; $lb_C(S_1) = ub_C(S_1) = lb_C(S_2) = ub_C(S_2) = 1$. BC on $SB + card$ fails whereas BC on *SB* alone does not prune.

To show that $(SB + card) \succ card$, take the constraint $S \supseteq \{2\}$ and universe $U = \{1,2,3\}$. Assume that $D(S) = \{\{1\},\{1,3\}\}$. The *SB* and *card* domains are: $lb_{SB}(S) = \{1\}$, $ub_{SB}(S) = \{1,3\}$; $lb_C(S) = 1$, $ub_C(S) = 2$. BC on $SB + card$ fails whereas BC on *card* alone does not prune. □

6 Generalising the Framework

Our framework for combining representations can be generalised to combinations of more than two representations. Thanks to this generalisation, we will be able to reason with, for instance, the hybrid domain representation [14] and the multiset representations of [9, 10]. The extension of the strong combination operator is straightforward as we can take the domain of S in $R_1 \times R_2$ to be

$D_{R_1R_2}(S)$. The extension of the weak combination operator is more problematic as the domain of S in $R_1 + R_2$ is not defined explicitly. This necessitates a generalization of the definitions of synchronization and bound consistency.

6.1 Combinations of Representations

Given n representations, R_1 to R_n, we write $D_{R_1 \ldots R_n}(S)$ for $D_{R_1}(S) \cap \ldots \cap D_{R_n}(S)$.

Definition 6 (Synchronization for $R_1 \times \ldots \times R_n$). *Given a variable S, the representation R_i is synchronized with $R_1 \times \ldots \times R_n$ on S iff:*

- $lb_{R_i}(S) = glb_{R_i}(D_{R_1 \ldots R_n}(S))$;
- $ub_{R_i}(S) = lub_{R_i}(D_{R_1 \ldots R_n}(S))$.

S is synchronized for $R_1 \times \ldots \times R_n$ iff $\forall i \in \{1, \ldots, n\}$, R_i is synchronized with $R_1 \times \ldots \times R_{i-1} \times R_{i+1} \times \ldots \times R_n$ on S.

Definition 7 (Synchronization for $R_1 + \ldots + R_n$). *Given a variable S, and the combination $(R_1 \times \ldots \times R_j) + (R_{j+1} \times \ldots \times R_n)$, we say that $R_1 \times \ldots \times R_j$ is synchronized with $R_{j+1} \times \ldots \times R_n$ on S iff S is synchronized for $R_1 \times \ldots \times R_j$ and $\forall i \in \{1, \ldots, j\}$, R_i is synchronized with $R_{j+1} \times \ldots \times R_n$ on S.*

S is synchronized for $(R_1 \times \ldots \times R_j) + \ldots + (R_k \times \ldots \times R_n)$ iff every pair of strong combinations is synchronized with every other on S.

Definition 8 (Bound consistency on $R_1 \times \ldots \times R_n$). *Given variables $S_1, \ldots, S_m$, and a constraint $c(S_1, \ldots, S_m)$, S_i is bound consistent on c from R_j to $R_1 \times \ldots \times R_{j-1} \times R_{j+1} \times \ldots \times R_n$ iff:*

- $lb_{R_j}(S) = glb_{R_j}(c[S_i]_{R_1 \ldots R_n})$;
- $ub_{R_j}(S) = lub_{R_j}(c[S_i]_{R_1 \ldots R_n})$;

S_i is bound consistent on c for $R_1 \times \ldots \times R_n$ iff S_i is synchronized for $R_1 \times \ldots \times R_n$ and $\forall j \in \{1, \ldots, n\}$, S_i is bound consistent from R_j to $R_1 \times \ldots \times R_{j-1} \times R_{j+1} \times \ldots \times R_n$.

Definition 9 (Bound consistency on $R_1 + \ldots + R_n$). *Given variables $S_1, \ldots, S_m$ and a constraint $c(S_1 \ldots, S_m)$, S_i is bound consistent on c for $(R_1 \times \ldots \times R_j) + \ldots + (R_k \times \ldots \times R_n)$ iff S_i is BC on c for every strong combination and S_i is synchronized for $(R_1 \times \ldots \times R_j) + \ldots + (R_k \times \ldots \times R_n)$.*

6.2 Comparison of Multiple Combinations

We give some algebraic identities that are useful in comparing (combined) representations. We first give a generalization of the result that $(R_1 \times R_2) \succeq (R_1 + R_2)$. Replacing a weak combination operator with a strong combination operator strengthens the representation.

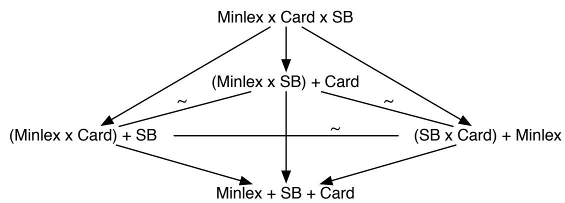

Fig. 2. Comparison of triple combinations with *minlex*, *SB* and *card*. $H_1 \rightarrow H_2$ means $H_1 \succ H_2$ whilst $H_1 \tilde{-} H_2$ means that $H_1 \sim H_2$.

Theorem 7. $(R_i \times \ldots \times R_k \times R_{k+1} \times \ldots \times R_n) \succeq (R_i \times \ldots \times R_k) + (R_{k+1} \times \ldots \times R_n)$.

We also give a generalization of the result that $(R_1 + R_2) \succeq R_1$ and $(R_1 + R_2) \succeq R_2$. Multiple weak combinations are stronger than their constituent parts.

Theorem 8. $(R_1 + \ldots + R_n) \succeq R_k$ *for all* $k \in \{1, \ldots, n\}$.

Finally, we give a generalization of the result that $(R_1 \times R_2) \succeq R_1$ and $(R_1 \times R_2) \succeq R_2$. Multiple strong combinations are stronger than their parts.

Theorem 9. $(R_1 \times \ldots \times R_n) \succeq R_k$ *for all* $k \in \{1, \ldots, n\}$.

6.3 Example

We illustrate this general framework with triple combinations of *minlex*, *SB*, and *card*. By using the weak and the strong combination operators in the ways permitted, we obtain 5 different combined representations: $minlex \times card \times SB$, $(minlex \times SB) + card$, $(minlex \times card) + SB$, $(SB \times card) + minlex$, and $minlex + SB + card$. The necessary synchronisation and BC rules can be obtained by instantiating Definitions 7 and 9. In addition, using the identities given in Theorem 7 to Theorem 9, we can compare the strength of these combinations and contrast with (combined) representations involving a subset of the three representations, i.e. $minlex \times SB$, $minlex \times card$, $SB \times card$, *minlex*, *SB*, and *card*. In Figure 2, we show how the triple combinations compare. Due to space, we prove here only one of the results.

Theorem 10. $(minlex \times SB) + card \sim (minlex \times card) + SB$.

Proof. We first show $(minlex \times card) + SB \not\succeq (minlex \times SB) + card$. Take the constraint $|S_1 \cap S_2| = 1$ on variables S_1 and S_2, and the universe $U = \{1, 2, 3, 4, 5\}$. Assume that $D(S_1) = D(S_2) = \{\{1, 5\}, \{3, 4\}\}$. The *minlex*, *SB* and *card* domains are:

- $D_{ML}(S_1) = D_{ML}(S_2) = \{\{1, 5\}, \{2\}, \{2, 3\}, \{2, 3, 4\}, \{2, 3, 4, 5\}, \{2, 3, 5\}, \{2, 4\}, \{2, 4, 5\}, \{2, 5\}, \{3\}, \{3, 4\}\}$;
- $lb_{SB}(S_1) = lb_{SB}(S_2) = \{\}$, $ub_{SB}(S_1) = ub_{SB}(S_2) = \{1, 3, 4, 5\}$;

– $lb_C(S_1) = lb_C(S_2) = ub_C(S_1) = ub_C(S_2) = 2$.

$(minlex \times SB) + card$ fails ($\{3\}$ is pruned thanks to synchronization with $card$) but $(minlex \times card) + SB$ does not prune.

We now show $(minlex \times SB) + card \not\succeq (minlex \times card) + SB$. Take the constraint $S_1 \cup S_2 \supseteq \{2, 3, 4\}$ on variables S_1 and S_2. Assume that $D(S_1) = D(S_2) = \{\{1, 2\}, \{1, 3\}, \{1, 4\}\}$. The *minlex*, *SB* and *card* domains are:

– $D_{ML}(S_1) = D_{ML}(S_2) = \{\{1, 2\}, \{1, 2, 3\}, \{1, 2, 3, 4\}, \{1, 2, 4\}, \{1, 3\}, \{1, 3, 4\}, \{1, 4\}\}$;
– $lb_{SB}(S_1) = lb_{SB}(S_2) = \{1\}, ub_{SB}(S_1) = ub_{SB}(S_2) = \{1, 2, 3, 4\}$;
– $lb_C(S_1) = lb_C(S_2) = ub_C(S_1) = ub_C(S_2) = 2$.

$(minlex \times card) + SB$ fails but $(minlex \times SB) + card$ does not prune. □

7 Related Work

Jefferson has provided in his Ph.D. thesis [6] a framework for comparing set representations and their combinations. Differently from [6], our framework allows the study of many different ways of combining multiple representations most of which were not previously known. In addition, the ordering we have defined on representations lets us compare the pruning power of the (combined) representations and consequently measure their ability to prune the search space under static variable ordering. [6] instead discusses search size comparison only briefly by showing representations which can represent more and produce smaller search trees. Altogether, in our work we can see what kind of combinations can be built, how they compare and which ones already exist or not. In the following, we show which of the existing combined representations are precisely situated by our framework.

7.1 Combination of two representations

The $SB + card$ representation is used in the Cardinal set solver [1]. The representation $SB \times card$ has been previously referred to as *subset-cardinality* [11] or *sbc*-domain [15]. A similar BC definition for $SB \times card$ is given in [15].

The representation *lengthlex*, although using both cardinality and lexicographic information, corresponds in our framework only to a single representation T defined by a total order. Given that both *lengthlex* and combinations of *minlex* and *card* representations focus on the same information, we might ask how they compare in general. Interestingly, the *lengthlex* representation, which simultaneously captures cardinality and lexicographic ordering information, is not better than two representations which deal with cardinality and lexicographic ordering information separately or simultaneously.

Theorem 11. *lengthlex is incomparable to minlex, minlex+card and minlex$\times$card.*

Proof. We first show $lengthlex \not\succeq minlex$. Consider a conjunction of a lexicographic constraint and two cardinality constraints, such as $(S_1 <_{lex} S_2 \wedge |S_1| = 2 \wedge |S_2| = 2)$. Assume $U = \{1,2,3,4\}$, $D(S_1) = \{\{2\},\{2,3\}, \{2,3,4\}\}$, $D(S_2) = \{\{1\},\{1,2\},\{1,3\},\{1,2,3\}\}$. The *minlex* and *lengthlex* domains are:

- $D_{ML}(S_1) = \{\{2\},\{2,3\},\{2,3,4\}\}$;
- $D_{ML}(S_2) = \{\{1\}, \{1,2\},\{1,2,3\},\{1,2,3,4\},\{1,2,4\},\{1,3\}\}$;
- $D_{LL}(S_1) = \{\{2\},\{3\},\{4\},\{1,2\},\{1,3\}, \{1,4\},\{2,3\},\{2,4\},\{3,4\},\{1,2,3\}, \{1,2,4\},\{1,3,4\},\{2,3,4\}\}$;
- $D_{LL}(S_2) = \{\{1\},\{2\},\{3\},\{4\},\{1,2\},\{1,3\},\{1,4\},\{2,3\},\{2,4\},\{3,4\},\{1,2,3\}\}$.

BC on *minlex* fails for both domains whilst BC on *lengthlex* does not fail: $D_{LL}(S_1) = \{\{1,2\},\{1,3\},\{1,4\},\{2,3\},\{2,4\}\}$ and $D_{LL}(S_2) = \{\{1,3\},\{1,4\},\{2,3\}, \{2,4\},\{3,4\}\}$ are BC. Thus, $lengthlex \not\succeq minlex$, and obviously we deduce $lengthlex \not\succeq (minlex + card)$ and $lengthlex \not\succeq (minlex \times card)$.

We now show $(minlex \times card) \not\succeq lengthlex$. Take again the constraint $(S_1 <_{lex} S_2 \wedge |S_1| = 2 \wedge |S_2| = 2)$ and $U = \{1,2,3,4\}$. Assume that $D(S_1) = \{\{2,3\},\{2,4\},\{3,4\},\{1,2,3\}\}$ and $D(S_2) = \{\{4\},\{1,2\},\{1,3\},\{1,4\}\}$. The *minlex*, *lengthlex* and *card* domains are:

- $D_{ML}(S_1) = \{\{1,2,3\},\{1,2,3,4\},\{1,2,4\},\{1,3\},\{1,3,4\},\{1,4\},\{2\},\{2,3\}, \{2,3,4\},\{2,4\},\{3\},\{3,4\}\}$;
- $D_{ML}(S_2) = \{\{1,2\},\{1,2,3\},\{1,2,3,4\},\{1,2,4\},\{1,3\},\{1,3,4\},\{1,4\},\{2\}, \{2,3\},\{2,3,4\},\{2,4\}, \{3\},\{3,4\},\{4\}\}$;
- $D_{LL}(S_1) = \{\{2,3\},\{2,4\},\{3,4\},\{1,2,3\}\}$;
- $D_{LL}(S_2) = \{\{4\}, \{1,2\},\{1,3\},\{1,4\}\}$;
- $lb_C(S_1) = 2$, $ub_C(S_1) = 3$;
- $lb_C(S_2) = 1$, $ub_C(S_2) = 2$.

With BC on $minlex \times card$, we obtain $D_{ML}(S_1) = \{\{1,3\},\{1,3,4\},\{1,4\},\{2\},\{2,3\}, \{2,3,4\},\{2,4\}\}$; $D_{ML}(S_2) = \{\{1,4\},\{2\},\{2,3\},\{2,3,4\},\{2,4\},\{3\},\{3,4\}\}$; $lb_C(S_1) = ub_C(S_1) = lb_C(S_2) = ub_C(S_2) = 2$. BC on *lengthlex* fails. Thus, $(minlex \times card) \not\succeq lengthlex$, and obviously we deduce $(minlex + card) \not\succeq lengthlex$ and $minlex \not\succeq lengthlex$. □

Even though the proof has focused on a conjunction of a lexciograhic constraint together with cardinality constraints, there are many other basic constraints, like subset, disjoint, atleast-k, and atmost-k, where *lengthlex* and the *minlex* and *card* combinations are incomparable. It is not difficult to prove that *lengthlex* is stronger than *card*. Figure 3 summarises these results.

7.2 Combination of three representations

In [11], a representation *length-lex** is introduced which corresponds to $lengthlex \times SB$ in our framework. In [15], the authors refer to this as the *ls*-domain; however they experiment with only $lengthlex + SB$. As $lengthlex \times SB$ and combinations of the *minlex*, SB and *card* represent similar information, we might ask how

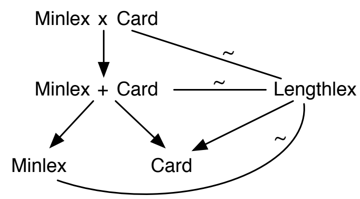

Fig. 3. Comparison of *minlex* and *card* combinations with *lengthlex*. $H_1 \rightarrow H_2$ means $H_1 \succ H_2$ whilst $H_1 \tilde{-} H_2$ means that $H_1 \sim H_2$.

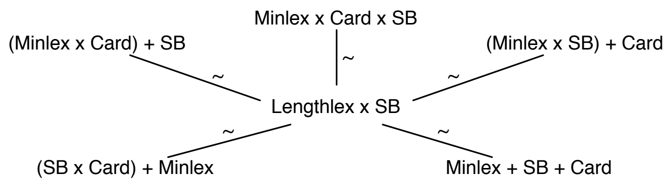

Fig. 4. Comparison of *minlex* based triplets against $lengthlex \times SB$. $H_1 \tilde{-} H_2$ means that $H_1 \sim H_2$.

they compare in general. In Figure 4, we compare the various combinations of *minlex*, *SB* and *card* against $lengthlex \times SB$. As we prove next, $lengthlex \times SB$ is incomparable to any *minlex* based combination, including *minlex* alone.

Theorem 12. *$lengthlex \times SB$ is incomparable with minlex, $minlex \times SB$, $minlex \times card$, $minlex + SB + card$, $(minlex \times card) + SB$, $(minlex \times SB) + card$, $(SB \times card) + minlex$ and $minlex \times card \times SB$.*

Proof. First we show that $(lengthlex \times SB) \not\succeq minlex$. Take the constraint $S \geq_{lex} \{1, 3\}$ on the variable S and the universe $U = \{1, 2, 3\}$. Assume that $D(S) = \{\{\}, \{1\}, \{1, 2\}, \{1, 2, 3\}\}$. The *lengthlex*, *SB* and *minlex* domains are:

- $D_{ML}(S) = \{\{\}, \{1\}, \{1, 2\}\{1, 2, 3\}\}$;
- $D_{LL}(S) = \{\{\}, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$;
- $lb_{SB}(S) = \{\}$, $ub_{SB}(S) = \{1, 2, 3\}$.

Enforcing BC for *minlex* fails whereas for $lengthlex \times SB$ it does not. Thus $lengthlex \times SB$ is not stronger than any combination of *minlex*.

We now show that $(minlex \times card \times SB) \not\succeq (lengthlex \times SB)$. Take the constraint $S_1 \cup S_2 = \{1, 2, 3, 4\}$ on two variables S_1 and S_2 and the universe $U = \{1, 2, 3, 4\}$. Assume that $D(S_1) = D(S_2) = \{\{3\}, \{4\}, \{1, 2\}\}$. The *minlex*, *lengthex*, *SB* and *card* domains are:

- $D_{LL}(S_1) = D_{LL}(S_2) = \{\{3\}, \{4\}, \{1, 2\}\}$;
- $lb_{SB}(S_1) = lb_{SB}(S_2) = \{\}$, $ub_{SB}(S_1) = ub_{SB}(S_2) = \{1, 2, 3, 4\}$;
- $D_{ML}(S_1) = D_{ML}(S_2) = \{\{1, 2\}, \{1, 2, 3\}, \{1, 2, 3, 4\}, \{1, 2, 4\}, \{1, 3\}, \{1, 3, 4\}, \{1, 4\}, \{2\}, \{2, 3\}, \{2, 3, 4\}, \{2, 4\}, \{3\}, \{3, 4\}, \{4\}\}$;

– $lb_C(S_1) = lb_C(S_1) = 1$, $ub_C(S_2) = ub_C(S_2) = 2$.

$lengthlex \times SB$ fails whereas $minlex \times card \times SB$ does not (note that the lower bound of *card* is updated to 2). Thus all *minlex* based combinations with SB and *card* are not stronger than $lengthlex \times SB$.

□

In [14], a combination of *maxlex*, SB and *card*, called a *hybrid domain*, is introduced and associated inference rules are given to maintain consistency between the bounds as well as to enforce some form of local consistency on common set constraints. By looking at these inference rules, we can see that this hybrid domain is a representation between $maxlex + SB + card$ and $maxlex \times SB \times card$. The reason is that two of the bounds in a hybrid domain are updated by looking at the other two at the same time, which is stronger reasoning than considering them pairwise as we do in $maxlex + SB + card$. However, reasoning about hybrid domain does not consider globally all the bounds at the same time. Consider for instance a variable S with the domain $lb_{maxlex}(S) = \{4, 2, 1\}$, $ub_{maxlex}(S) = \{4, 3, 1\}$, $lb_{SB}(S) = \{4\}$, $ub_{SB}(S) = \{1, 2, 3, 4\}$, $lb_C(S) = ub_C(S) = 3$. The inference rules will not change the subset bounds, but 1 must be in lb_{SB} for the three bounds to be synchronized together. Similary, consider a variable S with the domain $lb_{maxlex}(S) = \{4, 3\}$, $ub_{maxlex}(S) = \{5, 1\}$, $lb_{SB}(S) = \{\}$, $ub_{SB}(S) = \{1, 2, 3, 4, 5\}$, $lb_C(S) = ub_C(S) = 2$. The inference rules for hybrid domain will not change the subset bounds, but 2 must be removed from $ub_{SB}(S)$ for the three bounds to be synchronized together. Thus, hybrid domain provides between the weak and strong combination of its participating representations.

7.3 Combination of four representations

The authors propose in [9] a combined representation for multiset variables which corresponds to $SB + card + variety$ in our framework where *variety* is a representation based on the value of a multiset a set determined by its distinct number of elements $||a||$. Similar to *card*, in *variety* we have $D_V(S) = \{s \mid lb_V(S) \leq ||s|| \leq ub_V(S)\}$ and given a set of multisets $A \subseteq 2^U$, we define $glb_V(A) = \min(\{||a||, a \in A\})$ and $ub_V(A) = \max(\{||a||, a \in A\})$. In [10], this representation is expanded further by considering 8 different forms of lexicographic orderings on multisets. The authors compare theoretically these totally ordered representations in terms of expressiveness and compactness, and finally compare them experimentally wrt $SB + card + variety$. Our framework lets us conceive various ways of combining SB, *card*, *variety* and the totally ordered representations as well as compare their pruning power.

8 Conclusions

We have provided a general framework for combining together different representations of set and multiset variables. Within this framework, we have defined the notion of synchronization between representations. We have considered two different ways to propagate a combined representation: a weak method which considers each representation independently, and a strong method which considers them together. To compare representations, we have proposed an ordering on representations that measures their ability to prune the search space. Finally, we have undertaken a detailed study of combinations of representations. This identified several new relations. For example, the *lengthlex* representation, which simultaneously captures cardinality and lexicographic ordering information, is not better than a representation which deals with cardinality and lexicographic ordering information separately or simultaneously. As a second example, the hybrid domain of [14] provides between the weak and strong combination of the constituent representations. This paper opens doors to a number of interesting future work. First, we intend to study in depth different combined representations for multiset variables. Second, we want to explore what effects does combining representations has on the hardness of reaching fixed point [13]. Third, we will investigate efficient propagation algorithms for interesting combined representations and experiment with them.

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